Value Recursion in Monadic Computations

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Outline

- **Recursion and effects**
- Motivating examples
- Value recursion operators
- Properties
- The recursive do-notation
- Related work: How do we fit in?
- Summary, future work and conclusions

Recursion

- Two important uses of recursion:
  - As a control structure
    - Recursive functions
  - As a means for creating cyclic data structures
    - Streams, self referential data
- Semantics of recursion is well understood
  - Extensively studied since 60's
  - Modeled by least fixed-points

Effects

- Another fundamental programming technique
  - I/O is inevitable
  - Mutable variables, exceptions, non-determinism, ...
- Semantics of effects
  - Traditional semantics: Hoare logic
  - Monadic semantics: Moggi

The question

How do we model recursion in the presence of effects?

- Two different notions of recursion
  - The usual unfolding semantics
  - Value recursion

Unfolding recursion repeats effects

- $f :: \alpha \to \alpha, \qquad \mathit{fix}\ f = f\ (\mathit{fix}\ f)$
- Example:

$$\begin{aligned} &\mathit{fact}\ 0 = \mathit{return}\ 1 \\ &\mathit{fact}\ n = \mathbf{do}\ \mathit{putStrLn}\ (\text{“}\texttt{Now computing at: }\text{”} \mathbin{+\!\!+} \mathit{show}\ n) \\ &\qquad\qquad\quad r \leftarrow \mathit{fact}\ (n - 1) \\ &\qquad\qquad\quad \mathit{return}\ (n \times r) \end{aligned}$$

- Sample run

```
Main> fact 3
Now computing at: 3
Now computing at: 2
Now computing at: 1
6
```

Value recursion

- An alternative notion when recursion is only over *values*
  - The result of a monadic action is recursively defined
- Effects should neither be *lost* nor *duplicated* but *preserved*

$$
\begin{aligned}
\textbf{do}\ & w \leftarrow \ \ldots.\ x\ \ldots..\\
& x \leftarrow\ ..\ w\ ..\ x\ ..\\
& \ldots
\end{aligned}
$$

- Arises most frequently in embedded domain specific languages
  - Recursion in the meta-language is not sufficient to express recursion in the object-language

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Monadic GUI libraries

- Order determines screen layout:

$$\begin{array}{l} \textbf{do}\ \mathit{f1} \leftarrow \mathit{inputField}\ (\mathit{fieldSize}\ 10) \\ \quad\ \ \mathit{f2} \leftarrow \mathit{inputField}\ (\mathit{fieldSize}\ 10) \\ \quad\ \ \mathit{submitButton}\ (\mathit{someAction}\ \mathit{f1}\ \mathit{f2}) \\ \quad\ \ \ldots \end{array}$$

- What if submit button has to come first?

$$\begin{array}{l} \textbf{do}\ \mathit{submitButton}\ (\mathit{someAction}\ \mathit{f1}\ \mathit{f2}) \\ \quad\ \ \mathit{f1} \leftarrow \mathit{inputField}\ (\mathit{fieldSize}\ 10) \\ \quad\ \ \mathit{f2} \leftarrow \mathit{inputField}\ (\mathit{fieldSize}\ 10) \\ \quad\ \ \ldots \end{array}$$

Forking threads

- $forkIO :: IO\ () \rightarrow IO\ \ ThreadId$
- Run two algorithms on the same input, first one to finish kills the other

$$\begin{aligned} tryBoth\ \ inp\ = \mathbf{do}\ & t1 \leftarrow forkIO\ \ (alg1\ \ inp\ \ t2) \\ & t2 \leftarrow forkIO\ \ (alg2\ \ inp\ \ t1) \\ & \ldots \end{aligned}$$

$$\begin{aligned} alg1\ \ inp\ \ t\ \ = \mathbf{do}\ & ..\ compute\ \ with\ \ inp\ \ldots \\ & killThread\ \ t \end{aligned}$$

Modeling circuits using monads

- Lava, Hawk
- Multiple interpretations
- From the same description, just change the monad to
  - Simulate
  - Dump a netlist description
  - ...

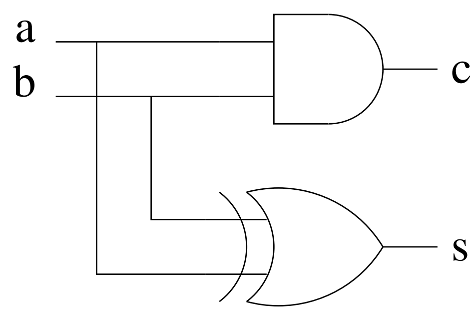
Basic idea

- Signals and Circuits, use abstraction:
  - *"Sig α"* to represent signals of type α
  - Monad *"C"* captures the underlying circuits semantics
- Basic components:

$$\begin{aligned}
&and &&:: Sig\ \ Bool \rightarrow Sig\ \ Bool \rightarrow C\ \ (Sig\ \ Bool)\\
&mux &&:: Sig\ \ Bool \rightarrow Sig\ \ \alpha \rightarrow Sig\ \ \alpha \rightarrow C\ \ (Sig\ \ \alpha)\\
&delay &&:: \alpha \rightarrow Sig\ \ \alpha \rightarrow C\ \ (Sig\ \ \alpha)
\end{aligned}$$

- *or*, *xor*, etc..

Half adder



$$halfAdd :: Sig\ Bool \rightarrow Sig\ Bool \rightarrow C\ (Sig\ Bool,\ Sig\ Bool)$$

$$\begin{aligned} halfAdd\ a\ b = \mathbf{do}\ & c \leftarrow and\ a\ b \\ & s \leftarrow xor\ a\ b \\ & return\ (c,\ s) \end{aligned}$$

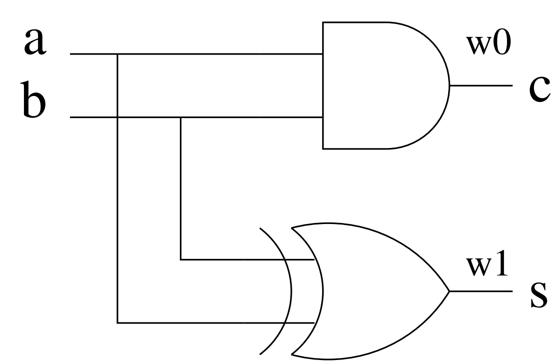
Using the half adder

- Simulation:

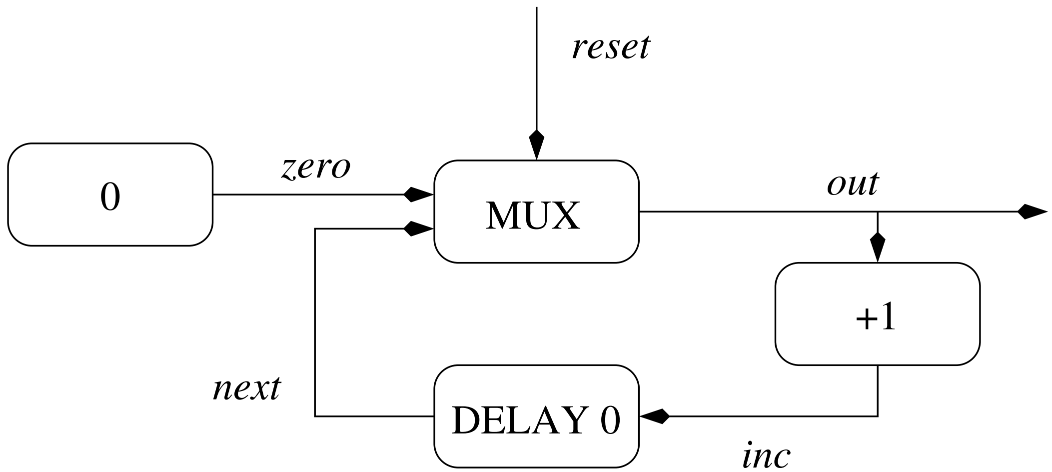
```
Main> halfAdd [True, True] [False, True]
([False,True],[True,False])
```

- NetList:

```
Main> halfAdd "a" "b"
w0 = and a b
w1 = xor a b
Result: (w0, w1)
```



Feedback in circuits



$$
\begin{aligned}
&counter && :: Sig\ Bool \rightarrow C\ (Sig\ Int) \\
&counter\ reset && = \mathbf{do}\ next \leftarrow delay\ 0\ inc \\
& && \qquad inc \leftarrow add1\ out \\
& && \qquad out \leftarrow mux\ reset\ zero\ next \\
& && \qquad zero \leftarrow constant\ 0 \\
& && \qquad return\ out
\end{aligned}
$$

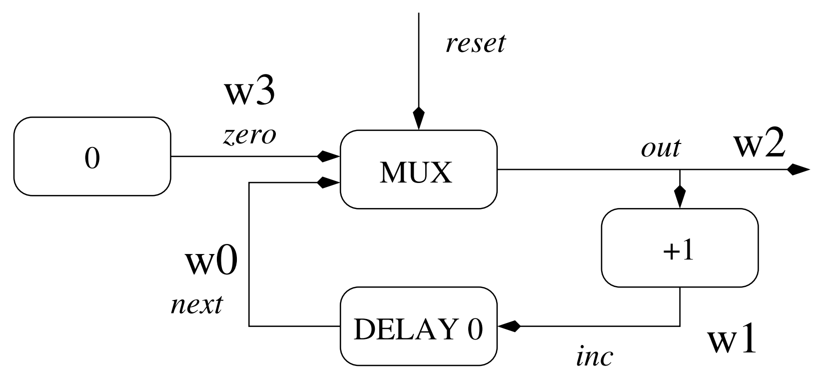
Using the counter

- Simulation:

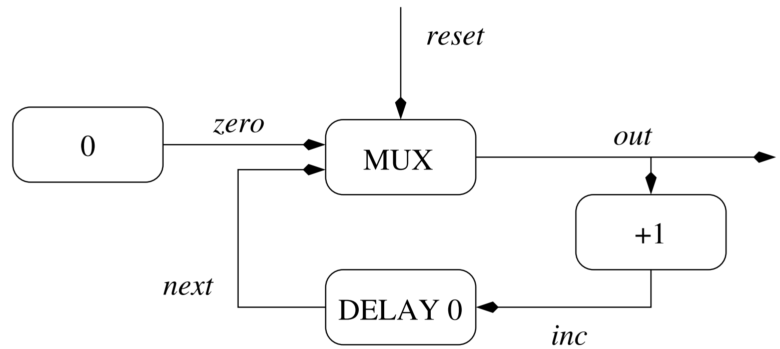
```
Main> counter [False, False, True, False, False, True, False]
[0,1,0,1,2,0,1]
```

- NetList:

```
Main> counter "reset"
w0 = delay 0 w1
w1 = add1 w2
w2 = mux reset w3 w0
w3 = constant 0
Result: w2
```



The problem



$$
\begin{array}{lll}
counter & :: & Sig\ Bool \rightarrow C\ (Sig\ Int) \\
counter\ reset & = & \mathbf{do}\ next \leftarrow delay\ 0\ inc \\
 & & \quad\ \ inc \leftarrow add1\ out \\
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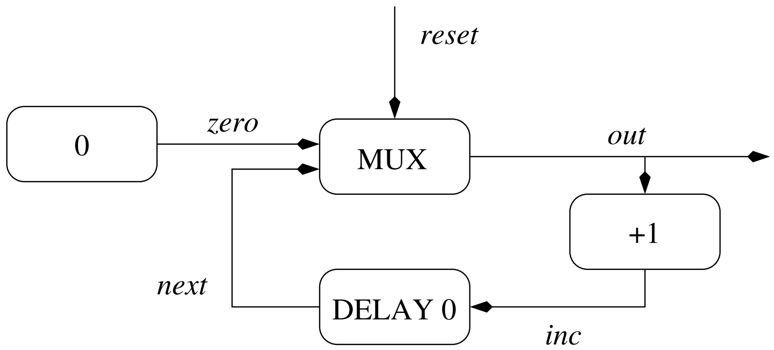
 & & \quad\ \ zero \leftarrow constant\ 0 \\
 & & \quad\ \ return\ out
\end{array}
$$

The problem



$$
\begin{aligned}
&counter :: Sig\ Bool \rightarrow C\ (Sig\ Int)\\
&counter\ reset = \mathbf{do}\ next \leftarrow delay\ 0\ inc\\
&\qquad inc \leftarrow add1\ out\\
&\qquad out \leftarrow mux\ reset\ zero\ next\\
&\qquad zero \leftarrow constant\ 0\\
&\qquad return\ out
\end{aligned}
$$

How to make the **do**-notation recursive?

Recursion at object-level and meta-level

Recursion in the meta-language is not sufficient to express recursion in the object-language

- Recall: usual recursion repeats effects
- We don't want circuit elements to be recreated by the fixed-point computation!
- Recursion is *only over the values*

Making the do-notation recursive

- Recall how recursive let works

$$\begin{array}{l} \textbf{let}\ x = 1 : y \\ \quad\ \ y = 2 : x \\ \textbf{in}\ x \end{array}$$

$$\begin{array}{l} \textbf{let}\ (x,\ y) = \\ \quad \mathit{fix}\ (\lambda(x,\ y).\ \textbf{let}\ x = 1 : y \\ \qquad\qquad\qquad\qquad\quad\ \ y = 2 : x \\ \qquad\qquad\qquad\qquad \textbf{in}\ (x,\ y)) \\ \textbf{in}\ x \end{array}$$

- Getting rid of recursive-let with *fix* and non-recursive let
- What if we have effects?

What we want

$$\begin{aligned}
&\mathit{mfix}\ (\lambda(\mathit{next},\ \mathit{inc},\ \mathit{out},\ \mathit{zero}).\\
&\qquad\qquad \mathbf{do}\ \mathit{next} \leftarrow \mathit{delay}\ 0\ \mathit{inc}\\
&\qquad\qquad\quad\ \ \mathit{inc} \leftarrow \mathit{add1}\ \mathit{out}\\
&\qquad\qquad\quad\ \ \mathit{out} \leftarrow \mathit{mux}\ \mathit{reset}\ \mathit{zero}\ \mathit{next}\\
&\qquad\qquad\quad\ \ \mathit{zero} \leftarrow \mathit{constant}\ 0\\
&\qquad\qquad\quad\ \ \mathit{return}\ (\mathit{next},\ \mathit{inc},\ \mathit{out},\ \mathit{zero})\\
&\quad)\\
&\ \gg\!= \lambda(\mathit{next},\ \mathit{inc},\ \mathit{out},\ \mathit{zero}).\ \mathit{return}\ \mathit{out}
\end{aligned}$$

- mfix: the *value recursion* operator

$$\mathit{mfix} :: (\alpha \to m\ \alpha) \to m\ \alpha$$

- Compare: $\mathit{fix} :: (\alpha \to \alpha) \to \alpha$

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Generalizing *fix*

- An *mfix* for all monads?
- Recall: $\mathit{fix}\ f\ =\ f\ (\mathit{fix}\ f), \quad \mathit{fix}\ ::\ (\alpha \to \alpha) \to \alpha$
- Possible definition for *mfix*:

$$\begin{aligned} \mathit{mfix} \quad &::\ (\alpha \to m\ \alpha) \to m\ \alpha \\ \mathit{mfix}\ f\ &=\ \mathit{mfix}\ f \gg\!= f \\ &=\ \mathit{fix}\ (\lambda m.\ m \gg\!= f) \end{aligned}$$

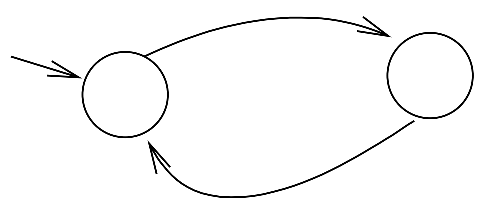
- $\mathit{fix}\, f = \bigsqcup \{\bot,\ f\ \bot,\ f\ (f\ \bot),\ f\ (f\ (f\ \bot)),\ ...\}$

mfix f = fix (λm. m $\gg=$ f)

- That is:

$$\begin{aligned} \mathit{mfix}\, f \;&=\; \bigsqcup \{\bot,\ \bot \gg= f,\ \bot \gg= f \gg= f, \\ &\qquad \bot \gg= f \gg= f \gg= f, \ldots\} \end{aligned}$$

- Would diverge whenever $\gg=$ is left-strict
- Computing the fixed point over both effects and values!
  - But we want to compute it only over the *values*



Example: State Monad

$$\textbf{type}\ \mathit{State} \quad = \ldots$$
$$\textbf{type}\ \mathit{ST}\ \alpha = \mathit{State} \to (\alpha,\ \mathit{State})$$

$$\begin{aligned}
\mathit{mfix} &:: (\alpha \to \mathit{ST}\ \alpha) \to \mathit{ST}\ \alpha \\
&:: (\alpha \to \mathit{State} \to (\alpha,\ \mathit{State})) \to \mathit{State} \to (\alpha,\ \mathit{State}) \\
\mathit{mfix}\ f &= \lambda s.\ \textbf{let}\ (a,\ s') = f\ a\ s \\
&\qquad\quad \textbf{in}\ (a,\ s')
\end{aligned}$$

Example: State Monad

$$\textbf{type}\ \mathit{State} \;=\; \ldots$$
$$\textbf{type}\ \mathit{ST}\ \alpha \;=\; \mathit{State} \rightarrow (\alpha,\ \mathit{State})$$

$$\begin{aligned}
\mathit{mfix} &:: (\alpha \rightarrow \mathit{ST}\ \alpha) \rightarrow \mathit{ST}\ \alpha \\
&:: (\alpha \rightarrow \mathit{State} \rightarrow (\alpha,\ \mathit{State})) \rightarrow \mathit{State} \rightarrow (\alpha,\ \mathit{State}) \\
\mathit{mfix}\ f &= \lambda s.\ \textbf{let}\ (a,\ s') = f\ a\ s \\
&\ \textbf{in}\ (a,\ s')
\end{aligned}$$

- State monad clearly separates values & effects
- Other monads are not that nice!
  - Maybe: $(\alpha \rightarrow \mathit{Maybe}\ \alpha) \rightarrow \mathit{Maybe}\ \alpha$
  - List: $(\alpha \rightarrow [\alpha]) \rightarrow [\alpha]$

Our Approach

- No generic solution, find individual instances
- Hypothesize expected properties of value recursion operators
- Study instances to verify properties
- Make a classification
  - identify important cases
  - identify cases when a recursive do-notation is feasible
- Relate these properties to those of *fix*

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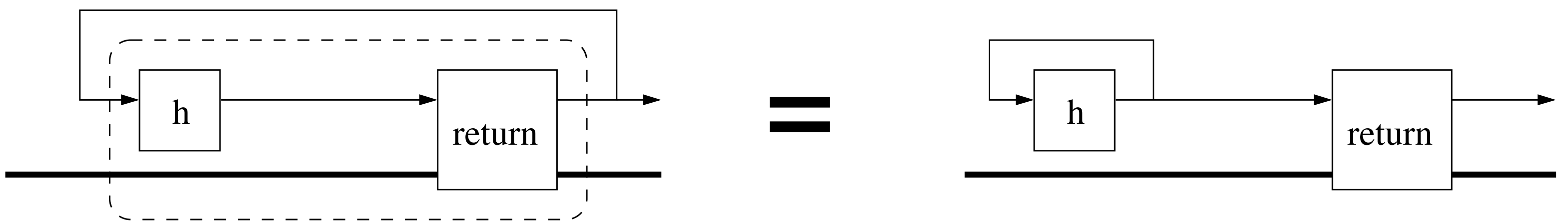
Basic Properties

- Strict functions
  - If f is a strict function, *mfix* f should be $\perp$
- Converse strictness
  - *mfix* f should be $\perp$ only when f is strict
- Purity
  - If there are no effects, *mfix* should behave just like *fix*

Purity



$$h :: \alpha \to \alpha$$

$$\mathit{mfix}\ (\mathit{return} \cdot h) = \mathit{return}\ (\mathit{fix}\ h)$$

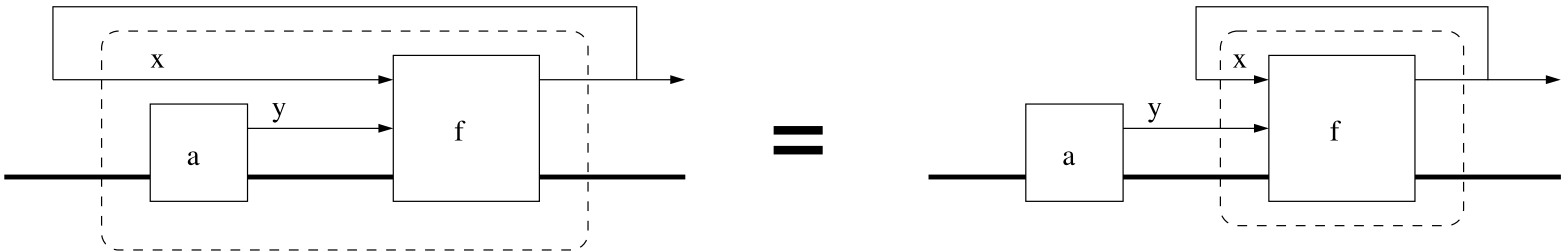
"Pure computations" have "pure fixed-points"

(If there are no effects, *mfix* is just *fix*)

Tightening properties

- Left tightening
  - A *preceeding* non-interfering computation can be pulled out of a recursive loop
- Right tightening
  - A *succeeding* non-interfering computation can be pulled out of a recursive loop

Left tightening



$a :: m\ \tau \qquad\qquad f :: \sigma \to \tau \to m\ \sigma$

$$mfix\ (\lambda x.\ a \gg\!= \lambda y.\ f\ x\ y) = a \gg\!= \lambda y.\ mfix\ (\lambda x.\ f\ x\ y)$$

Pulling a non-interfering computation out of the recursive loop: Tighten the loop from left

Yet others...

- Nesting property
  - Simultaneous and pointwise fixed points coincide
- Parametricity properties, if s is strict:
  - $g \cdot s = map\ s \cdot f \ \rightarrow map\ s\ (mfix\ f) = mfix\ g$
  - Similar to: $g \cdot s = s \cdot f \rightarrow s\ (fix\ f) = fix\ g$
- Sliding:
  - $mfix\ (map\ h \cdot f) = map\ h\ (mfix\ (f \cdot h))$
  - Similar to: $fix\ (h \cdot f) = h\ (fix\ (f \cdot h))$
- etc...

		Strict	Pure	Left	Nest	Slide	Right
Identity		✔	✔	✔	✔	✔	✔
Maybe		✔	✔	✔	✔	✗	✗
Lists		✔	✔	✔	✔	✗	✗
State	$mfix_0$	✔	✔	✔	✔	✗	✗
	$mfix_i$	✔	✔	✔	✗	✗	✗
	$mfix_\omega$	✔	✔	✔	✔	✔	✔
Output		✔	✔	✔	✔	✔	✔
Environment		✔	✔	✔	✔	✔	✔
Continuations		?					

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mdo-notation

- Compare:

$$
\begin{array}{l}
\mathit{mfix}\ (\lambda(\mathit{next},\ \mathit{inc},\ \mathit{out},\ \mathit{zero}).\\
\qquad\qquad \mathbf{do}\ \begin{array}{lll}
\mathit{next} & \leftarrow & \mathit{delay}\ 0\ \mathit{inc}\\
\mathit{inc} & \leftarrow & \mathit{add1}\ \mathit{out}\\
\mathit{out} & \leftarrow & \mathit{mux}\ \mathit{reset}\ \mathit{zero}\ \mathit{next}\\
\mathit{zero} & \leftarrow & \mathit{constant}\ 0\\
\multicolumn{3}{l}{\mathit{return}\ (\mathit{next},\ \mathit{inc},\ \mathit{out},\ \mathit{zero})}
\end{array}\\
\quad)\\
\ \gg\!=\ \lambda(\mathit{next},\ \mathit{inc},\ \mathit{out},\ \mathit{zero}).\ \mathit{return}\ \mathit{out}
\end{array}
$$

mdo-notation (cont.)

- To:

$$
\begin{aligned}
\textbf{do}\ & next \leftarrow delay\ 0\ \ inc \\
& inc \quad \leftarrow add1\ \ out \\
& out \quad \leftarrow mux\ \ reset\ \ zero\ \ next \\
& zero \leftarrow constant\ 0 \\
& return\ (next,\ out)
\end{aligned}
$$

The *MonadRec* class

$$\textbf{class}\ Monad\ m \Rightarrow MonadRec\ m\ \textbf{where}$$
$$\mathit{mfix} :: (\alpha \to m\ \alpha) \to m\ \alpha$$

- **mdo** to be available for all instances of *MonadRec*

Importance of Left Tightening

$$
\begin{array}{lll}
\textbf{mdo}\ x \leftarrow e_1 & & \textbf{do}\ x \leftarrow e_1 \\
\quad\quad\ \ y \leftarrow e_2 & \Longrightarrow & \quad\ \ \textbf{mdo}\ y \leftarrow e_2 \\
\quad\quad\ \ e_3 & & \quad\quad\quad\quad\ \ e_3
\end{array}
$$

- $x, y \notin FV(e_1)$
- If there is no recursion, *mfix* has no effect!
  - **mdo** is the same as **do** in that case
  - Backward compatibility

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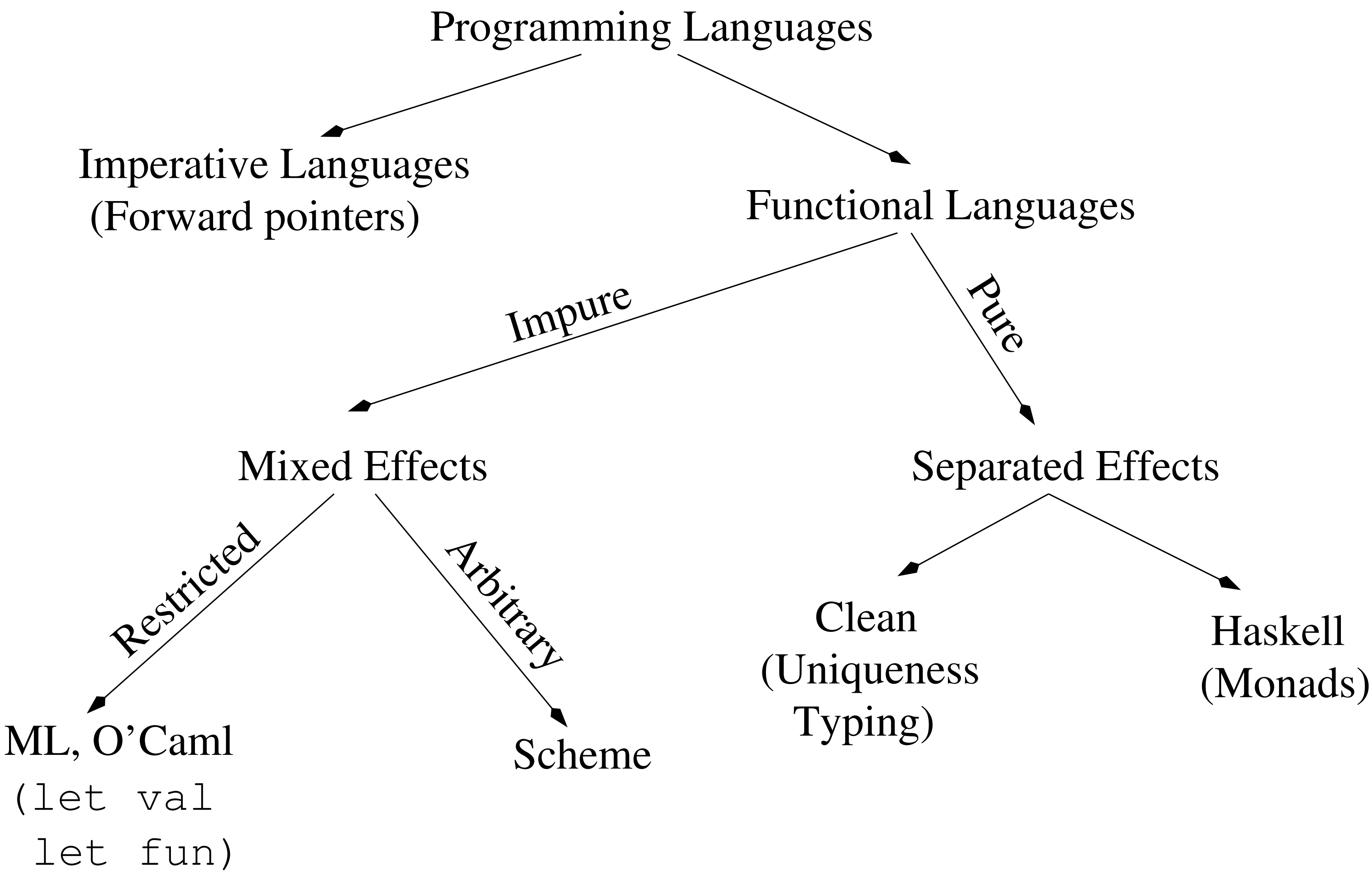
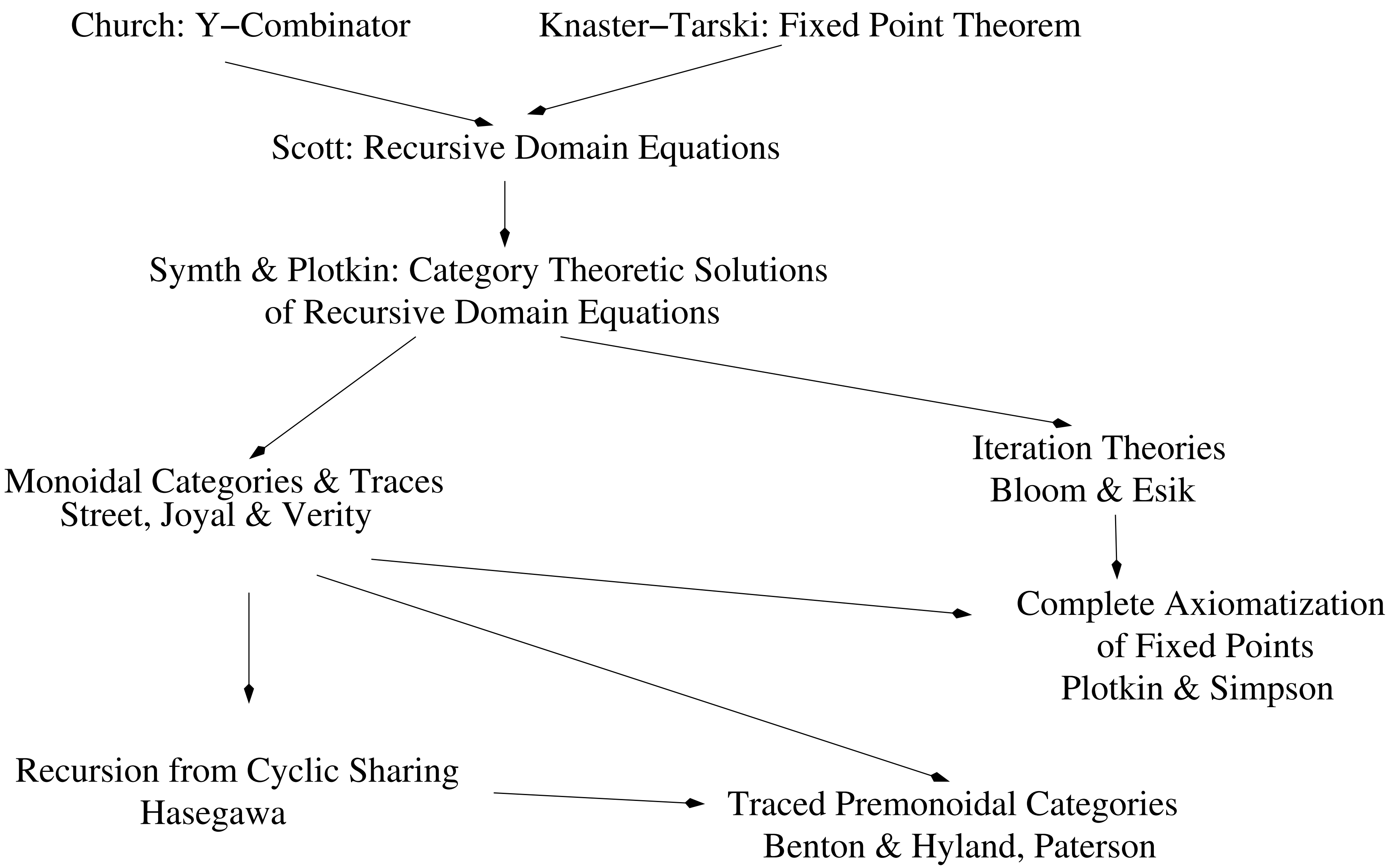
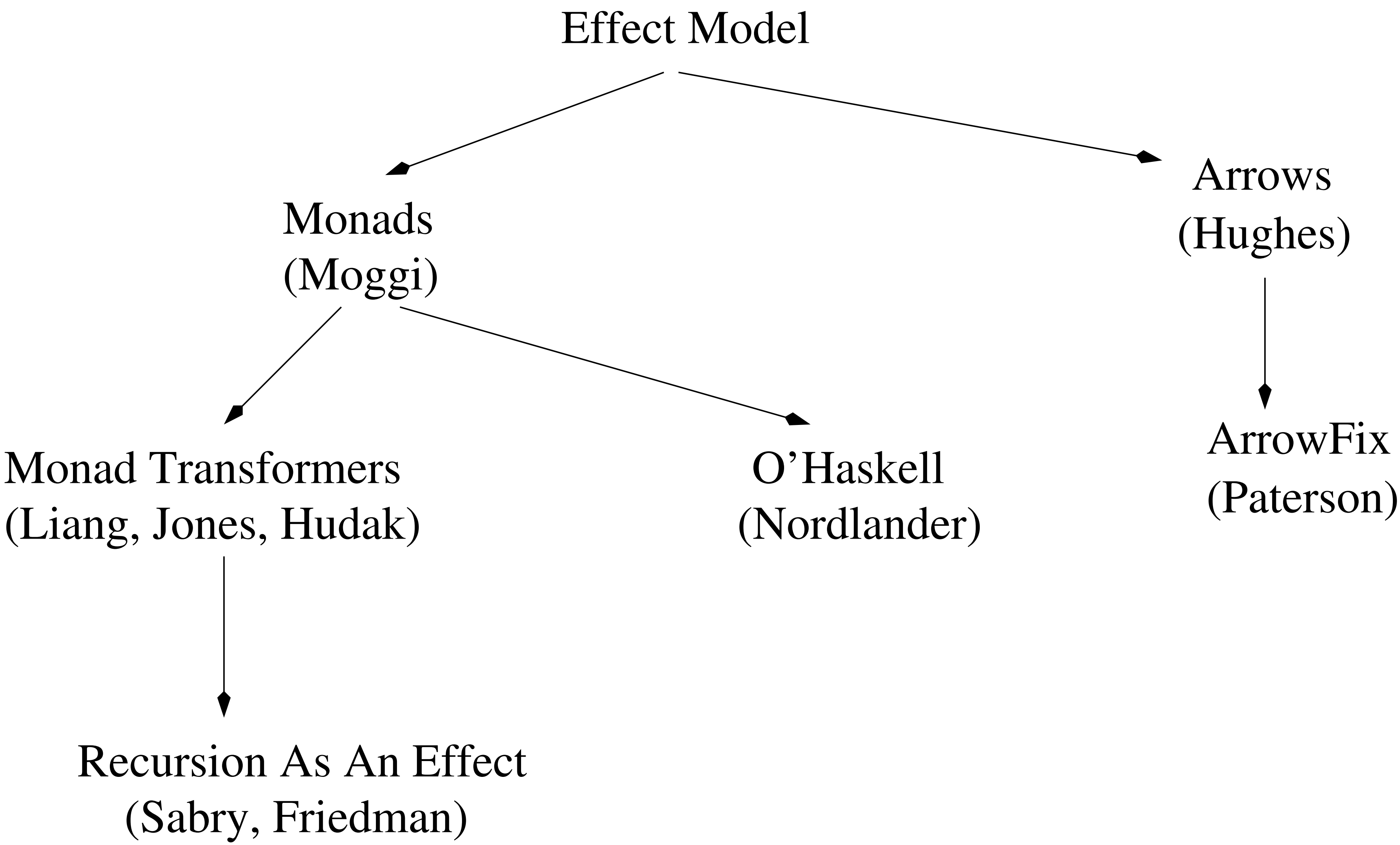
Effects


Programming Languages
Imperative Languages
(Forward pointers)
Functional Languages
Impure
Pure
Mixed Effects
Separated Effects
Restricted
Arbitrary
ML, O'Caml
(let val
let fun)
Scheme
Clean
(Uniqueness
Typing)
Haskell
(Monads)

Fixed-points



Value Recursion



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Summary

- Search for a generic *mfix*
- Properties, both expected and derived
- Study of monads
  - Identity, exceptions (maybe), non-determinism (list), state, environment, output, trees, fudgets, I/O ...
- Embeddings
  - Preservation of properties through embeddings of monads

Summary (cont.)

- Transformers
  - Obtaining a new *mfix* by transforming an old one
- The mdo-notation
  - Typing
  - Pragmatics
    * Repeated variables
    * Let-generators (monomorphic)
  - Translation algorithm
  - Implementation in February 2001 release of Hugs

Summary (cont.)

- The IO monad and *fixIO*
  - Two level semantics
    - Top layer handles "functional" core
    - Bottom layer handles I/O
    - Clear interaction via reduction rules
  - Operational meaning of *fixIO* clarified

Summary (cont.)

- Relation to other axiomatizations
  - *arrowFix*
  - "traced premonoidal categories"
  - They are cleaner, but limited applicability
    - **OK:** State (lazy), environment, output
    - **But not:** Exceptions, lists, strict state, IO, tree, fudgets, ...

Summary (cont.)

- Examples, case studies
  - Circuit simulation
  - Bird's *replaceMin* problem
  - Sorting networks, GUI layout problem
  - Interpreters
  - Doubly-linked circular lists with stateful nodes
  - Logical variables

Future work

- Practical:
  - Support for **mdo** in all Haskell systems
  - Opportunities in other paradigms
  - More monads...
- Theoretical:
  - Semantics of *fixIO* needs more work (parametricity)
  - A more precise "categorical" account via traces
  - A precise analysis for the continuation monad

Conclusions

- Theory: value recursion operators form an interesting class
  - Making the interaction between effects and recursion clear is important
- Practice: Work on **mdo** provides necessary syntactic support in Haskell
  - Lava and Hawk can really use it
  - More in the spirit of Haskell:

    let is recursive, why not **do**?

Conclusions (cont.)

- Future of functional programming
  - Lazy imperative programming
  - Semantics and implementation of embeded domain specific languages
  - Multiple interpretations

All heavily rely on monads, and recursion is inevitable