Recursive Monadic Bindings

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ABSTRACT

Monads have become a popular tool for dealing with computational effects in Haskell for two significant reasons: equational reasoning is retained even in the presence of effects; and program modularity is enhanced by hiding "plumbing" issues inside the monadic infrastructure. Unfortunately, not all the facilities provided by the underlying language are readily available for monadic computations. In particular, while recursive monadic computations can be defined directly using Haskell's built-in recursion capabilities, there is no natural way to express recursion over the *values* of monadic actions. Using examples, we illustrate why this is a problem, and we propose an extension to Haskell's do-notation to remedy the situation. It turns out that the structure of monadic value-recursion depends on the structure of the underlying monad. We propose an axiomatization of the recursion operation and provide a catalogue of definitions that satisfy our criteria.

1. INTRODUCTION

We begin with a puzzle. Consider the following piece of almost-Haskell code:

```
isEven :: Int -> Maybe Int
isEven n = if even n then Just n else Nothing

puzzle :: [Int]
puzzle = do (x, z) <- [(y, 1), (y^2, 2), (y^3, 3)]
            Just y <- map isEven [z+1 .. 2*z]
            return (x + y)
```

Notice that variable y appears free in the first line of the do-expression: for the sake of this puzzle, assume that the do-notation binds variables recursively, much like `let` of Haskell or `letrec` of Scheme. Under this assumption, what should the value of `puzzle` be?

Our goal in this paper is to provide a general answer to this type of question. We develop a framework for recursion over the values resulting from monadic actions. The discussion is set in the context of Haskell, but the ideas have wider applicability.

We first motivate the need for recursion in monadic computations, then we propose an extension to the do-notation supporting recursive bindings. We proceed to argue that the structure of the underlying monad specifies how the recursion should be performed and axiomatize the required behavior. The remainder of the paper contains a catalogue of monads that have recursion operators satisfying our criteria. On the way, of course, we provide an answer to the puzzle (in Section 6.3).

2. MOTIVATING PROBLEMS

In this section, we present two examples to motivate the value of having recursive bindings in the do-notation. The first example is about sorting-networks with traces and is done in some detail. The second is from modeling circuits. We cover this much more briefly.

2.1 Sorting networks

A *sorting network* is a collection of comparators, connected in such a way that the output of the network is always the sorted permutation of its input [2]. Figure 1 shows an example that can sort four numbers. For each comparator, the wire to its right carries the maximum of its inputs, while the lower one carries the minimum.

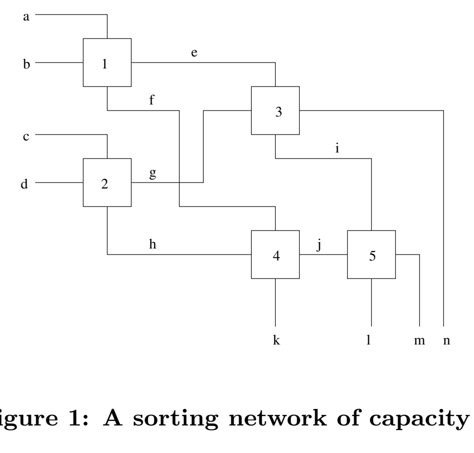

Figure 1: A sorting network of capacity 4

In this particular example, a, b, c, and d are the inputs, while k, l, m, and n are the outputs.

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How can we implement a sorting network so that we not only get the values sorted, but also a transcript of the operations performed during sorting? We want each comparator unit to report on the operation it performed while sorting took place. The *output* monad springs to mind. We translate the sorting network in Figure 1 almost literally into Haskell code of Figure 2.

```
newtype Out a = Out (a, String)

instance Monad Out where
  return x          = Out (x, "")
  Out ~(x, s) >>= f = let Out (y, s') = f x
                      in Out (y, s ++ s')

instance Show a => Show (Out a) where
  show (Out (v, s))  = "Value: " ++ show v
                         ++ "\nTrace:" ++ s

comp :: Int -> (Int, Int) -> Out (Int, Int)
comp i (a, b) = Out ((max a b, min a b), msg)
      where c1  = ": swap: " ++ show (a, b)
            c2  = ": pass: " ++ show (a, b)
            msg = "\nUnit " ++ show i ++
                       (if a < b then c1 else c2)

type QuadInts = (Int, Int, Int, Int)
sort4 :: QuadInts -> Out QuadInts
sort4 (a, b, c, d) =
         do (e, f) <- comp 1 (a, b)  -- unit 1
            (g, h) <- comp 2 (c, d)  -- unit 2
            (n, i) <- comp 3 (e, g)  -- unit 3
            (j, k) <- comp 4 (f, h)  -- unit 4
            (m, l) <- comp 5 (i, j)  -- unit 5
            return (k, l, m, n)
```

Figure 2: Haskell code for the network in Figure 1

Here is a sample run:

```
Main> sort4 (23, 12, -1, 2)
Value: (-1,2,12,23)
Trace:
Unit 1: pass: (23,12)
Unit 2: swap: (-1,2)
Unit 3: pass: (23,2)
Unit 4: pass: (12,-1)
Unit 5: swap: (2,12)
```

A quick look at the trace reveals that it is consistent with the operation of the network for this particular input.

In the definition of `sort4`, we carefully selected the execution order of the units such that all values were available before they were used. What if it was inconvenient to arrange for this? In our example, for instance, what if we want to see the output of unit 3 after unit 5? Notice that unit 5 uses the value i, which is produced by unit 3. Ideally, we would like to be able to change the function `sort4` to:

```
sort4 (a, b, c, d) =
          do (e, f) <- comp 1 (a, b)  -- unit 1
             (g, h) <- comp 2 (c, d)  -- unit 2
             (j, k) <- comp 4 (f, h)  -- unit 4
             (m, l) <- comp 5 (i, j)  -- unit 5
             (n, i) <- comp 3 (e, g)  -- unit 3
             return (k, l, m, n)
```

That is, we move the line corresponding to unit 3 after that of unit 5. Although this is the most intuitive thing to do, the resulting code is no longer valid: The variable i is not in scope when it's used.

How should we fix this? In this simple case, the obvious solution would be to postprocess the output of the original program to obtain the required ordering. But this is not a very satisfactory approach. In particular, the failed attempt of reordering lines in the do-expression is quite appealing. After all, the value that is computed by `sort4` (i.e. the quadruple representing the sorted permutation) does *not* depend on the order we observe the output. The attempt would have been successful, if only we had a way to bind variables recursively in the do-notation.

2.2 Resettable counter

The previous example didn't absolutely require recursive bindings because the values bound by the do-notation could be sequentially defined. This is not always the case. Our second example comes from the hardware-modeling domain. Microarchitectural design languages have been the target of programming language research in recent years because of the complexity of such designs. Lava [1] and Hawk [11, 16] are two recent systems designed to address this need. Lava uses monads intensively in modeling various circuit elements. Originally, Hawk used a similar monad-based approach as well. This technique provides a very flexible framework for translating specifications to VHDL or Verilog descriptions that could be used in producing real circuits. By just "plugging-in" the appropriate monad, the very same description can be used in simulation or in obtaining descriptions of the circuit in other languages. But this comes at a certain cost, as we explore here.

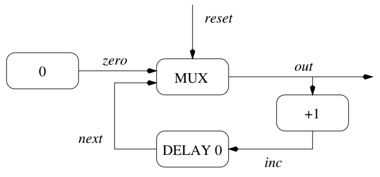


Figure 3: Resettable counter circuit

Hawk uses lazy lists to model signals flowing through circuits. The early monad-based implementation of Hawk used a `Circuit` monad, which is basically a combination of the state and output monads. Without going into details, we consider the circuit of Figure 3 as modeled in Hawk. Using the `Circuit` monad, we would like to model this circuit by:

```
counter :: Signal Bool -> Circuit (Signal Int)
counter reset = do next <- delay 0 inc
                   inc  <- lift1 (+1) out
                   out  <- mux reset zero next
                   zero <- lift0 0
                   return out
```

Notice that the description follows the circuit almost literally, but again, the program presented is not valid Haskell. The variables *inc, out* and *zero* are used before they are defined. Furthermore, because the definitions are cyclic, there is no way to serialize this program. The feedback present in the circuit causes the problem. Again, we need to be able to bind variables recursively in the do-notation. This problem is the main reason why the current implementation of Hawk does not use explicit monadic style.

3. RECURSIVE BINDINGS FOR DO

Currently, a do-expression in Haskell behaves like the `let*` of Scheme: the bound variables are available only in the textually following expressions. We need the do-notation to behave more like the `let` of Haskell, which allow recursive bindings. Of course, it is not necessarily the case that all monads will allow for such recursive bindings. We call a monad *recursive*, if there is a "sensible" way to allow for this kind of recursion. We codify what "sensible" should mean in Section 4. In this section, we look at a syntactic extension to Haskell that allows recursive bindings in the do-notation. This extension is a variant of the do-notation, called the μdo-notation. Just like the do-notation is available for any monad, the μdo-notation will be automatically available for any recursive-monad.

3.1 The new translation

Recall that a do-expression is translated into a series of applications of $\gg\!=$ [10]. Similarly, we need μdo to translate into more primitive components. We use a fixed-point operator, called mfix, whose type is $\forall a.\ (a \to m\ a) \to m\ a$, where m is the underlying monad. The translation is:

$$
\begin{array}{lcl}
 & & \texttt{mfix}\ (\lambda\tilde{}BV.\ \texttt{do}\ p_1 \leftarrow e_1 \\
\mu\texttt{do}\ p_1 \leftarrow e_1 & & \qquad\qquad \ldots \\
\quad \ldots & \Longrightarrow & \qquad\qquad p_n \leftarrow e_n \\
\quad p_n \leftarrow e_n & & \qquad\qquad v \leftarrow e \\
\quad e & & \qquad\qquad \texttt{return}\ BV) \\
 & & \quad \gg\!=\ \lambda\ BV.\ \texttt{return}\ v
\end{array}
$$

where BV stands for the k-tuple consisting of all the variables occurring in all the binding patterns plus the brand new variable v. Notice that each one of $p_1 \ldots p_n$, the binding patterns, can be any valid Haskell pattern, not just simple variables. The variables that are bound by these patterns may appear anywhere in $e_1 \ldots e_n$ and e. A variable may not be multiply bound: neither in the same pattern, nor in different patterns.

As an example, consider the following μdo expression which implements a sorting network for three numbers:

```
mdo (d, e) <- comp 1 (a, b)
    (i, h) <- comp 3 (d, f)
    (f, g) <- comp 2 (e, c)
    return (g, h, i)
```

After the translation, it becomes:

```
mfix (\~(d, e, i, h, f, g, v) ->
     do (d, e) <- comp 1 (a, b)
        (i, h) <- comp 3 (d, f)
        (f, g) <- comp 2 (e, c)
        v      <- return (g, h, i)
        return (d, e, i, h, f, g, v))
   >>= \(d, e, i, h, f, g, v) -> return v
```

The instance of mfix used in the translation is automatically deduced by the Haskell type system to be the instance at the output monad.

Although the translation of μdo using *do* is similar to the translation of `letrec` using `let` and the usual fixed-point operator fix, there is an important issue. Languages such as Haskell provide a generic definition of fix that works for all types. But there seems to be no appropriate generic definition of mfix that will work for all monads. Instead, we have to find an appropriate definition of mfix for each monad. To achieve some level of of uniformity, we stipulate some axioms that mfix must satisfy, and attempt to discover satisfactory definitions for individual monads. We say that a monad is *recursive* when there is a definition of mfix satisfying our axioms. A μdo-expression is well-typed if the underlying monad is recursive and the translation is well-typed.

3.2 Repeated variables and let-bindings

In the translation of the μdo-notation, we explicitly prohibited a variable from being repeated in different patterns (repetition within the same pattern is disallowed following the usual Haskell convention). In the do-notation, a repeated variable has nothing to do with its previous binding: a new binding using the same name shadows the earlier one. If we allow repetitions in the μdo-notation, however, the translation would not treat them as independent. Furthermore, a repeated variable might change its type in the do-notation, but this will fail to type check for the μdo-notation. More importantly, one might expect that repeated variables will provide a way of constraining the values that they might take in the μdo-notation, which is not what the translation implies. Hence, even if the translation goes through, this might lead to misunderstandings.[1]

Allowing `let` bindings in the μdo-notation is another issue. In the do-notation, `let` bindings allow giving names to non-monadic computations in a convenient manner. Can we allow them in the μdo-notation as well? An obvious extension is to treat them as recursive bindings that are valid throughout the whole body, suggesting the following translation:

$$
\begin{array}{lcl}
 & & \texttt{mfix}\ (\lambda\ \tilde{}BV.\ \texttt{do}\ \texttt{...1...} \\
\mu do\ \texttt{...1...} & & \qquad\qquad \texttt{let}\ p_1\ \texttt{=}\ e_1 \\
\quad \texttt{let}\ p_1\ \texttt{=}\ e_1 & & \qquad\qquad\quad \ldots \\
\qquad \ldots & \Longrightarrow & \qquad\qquad\quad p_n\ \texttt{=}\ e_n \\
\qquad p_n\ \texttt{=}\ e_n & & \qquad\qquad \texttt{...2...} \\
\quad \texttt{...2...} & & \qquad\qquad v \leftarrow e \\
\quad e & & \qquad\qquad \texttt{return}\ BV) \\
 & & \quad \gg\!=\ \lambda\ BV.\ \ \texttt{return}\ v
\end{array}
$$

The translation is similar to what we had before, except now the variables bound in $p_1 \ldots p_n$ appear in BV as well. However, this poses some problems. In Haskell, `let` bound variables are polymorphic, while λ-bound ones are monomorphic. This implies that the variables bound in $p_1 \ldots p_n$ are monomorphic in the code block marked by `...1...` but polymorphic in $e_1 \ldots e_n$, `...2...`, and in e. This is not

[1]In a similar vein, the usual do-notation should not allow repetitions either. List comprehensions become especially horrible: `f x = [x | x <- [x..4], x <- [x..8]]` is a confusing (yet legal) Haskell function.

a desirable situation. As a concrete example consider the following translation:

```
                                mfix (\(~(z, y, f, v)) ->
 mdo                                do z <- return (f 2)
  z <- return (f 2)                    y <- return (f 'a')
  y <- return (f 'a')                  let f x = x
  let f x = x            ===>          v <- return ()
  return ()                            return (z, y, f, v))
                                >>= \(z, y, f, v) -> return v
```

The translation fails to type check for obvious reasons: The function f is no longer polymorphic.

The solution we adopt is to require let bindings to be monomorphic in a μdo. That is, `let` becomes just a syntactic sugar within μdo, translated as:[2]

$$\begin{array}{lcl} \texttt{let } p_1 \texttt{ = } e_1 & & p_1 \leftarrow \texttt{return } e_1 \\ \quad \ldots & \Longrightarrow & \ldots \\ \quad p_n \texttt{ = } e_n & & p_n \leftarrow \texttt{return } e_n \end{array}$$

This gives us a uniform design. If a polymorphic value definition is required, one should use the standard `let` expressions of Haskell, rather than the `let` generator, which will create its own scope with polymorphic names. (The translation and the related issues are discussed in detail in a companion implementation paper [4].)

3.3 Implementation

We have a modified version of Hugs supporting the μdo-notation.[3] This implementation acts as a preprocessor, i.e. it performs the translation at the source level, and hence the amount of modifications we made in the Hugs source code is fairly small. We expect the same to hold when the translation is done inside the compiler. Basically, the required changes will be localized to type checking and de-sugaring routines.

The related class declaration for recursive monads is:

```
  class Monad m => MonadRec m where
     mfix :: (a -> m a) -> m a
```

In this simple implementation, occurrences of `let` expressions are translated blindly, without requiring them to be monomorphic. The associated typing problems are ignored.

4. RECURSIVE MONADS

The previous section addressed syntax. Now we turn to the meat of the issue and study mfix directly. We start by looking for a generic mfix.

[2]This extends to functions as well, basically `let f x y = z` will become `f` $\leftarrow$ `return` $(\lambda x.\lambda y.$ `z`$)$.

[3]More information and downloading instructions are available at `http://www.cse.ogi.edu/PacSoft/projects/muHugs`.

4.1 The generic mfix

The fixed point operator, fix, which has type $\forall a.\ (a \to a) \to a$, has a generic definition that works for all types.[4] For a lazy language like Haskell, the definition is just:

$$\begin{array}{rcl} \text{fix} & :: & (a \to a) \to a \\ \text{fix } f & = & f\ (\text{fix } f) \end{array}$$

Is there a generic definition for mfix as well? Inspired by the generic definition of fix, we consider the following (all equivalent) definitions for mfix:

$$\begin{array}{rcl} \text{mfix} & :: & \text{Monad } m \Rightarrow (a \to m\ a) \to m\ a \\ \text{mfix } f & = & \text{mfix } f \gg\!= f \\ \text{mfix } f & = & \text{do } \{x \leftarrow \text{mfix } f; f\ x\} \\ \text{mfix } f & = & \text{fix } (\text{join} \cdot \text{map } f) \end{array}$$

Unfortunately, this definition is simply not appropriate. To see why, we should specify what sort of properties we want mfix to have. First of all, we would expect a constant function, one that ignores its argument and always returns the same result, should have that result as its fixed point. This certainly holds for fix. We illustrate that this property does not hold for the `Maybe` monad with this definition. Here is the simplest test:

```
Main> mfix (const (Just 3))
ERROR: Control stack overflow
```

It is not hard to see why this definition fails to satisfy the required property: Consider the third version of the attempted definition. Since both `join` and `map f` are strict, so is their composition: Since the least fixed-point of any strict function is $\perp$, the result is $\perp$ as well.

Looking closely at the default definition, we see the following: To compute the mfix of a function of type $a \to m\ a$, we first construct a function $m\ a \to m\ a$[5], and then compute the usual fixed-point of it. In other words, the fixed-point is computed not only for the values that are manipulated, but also for the effects that take place during the execution.

Now, recalling the original intuition behind μdo-notation, we see that this is not what we wanted. We want the fixed-point computation to take place *only* over the values of monadic actions, while the effects and other computations remain untouched.

4.2 Axiomatizing mfix

So far, we have been using phrases like "a suitable definition of mfix" somewhat loosely. The time has come to make "suitable" precise. We give three axioms that mfix must satisfy, summarized in Figure 4.

Axiom 1 is about pure computations:

$$\text{mfix } (\text{return} \cdot h) = \text{return } (\text{fix } h)$$

[4]Technically, the underlying type needs to be a pointed CPO, but this requirement is vacuously satisfied in Haskell as all types are pointed, i.e. non-termination can happen at any type.

[5]This is the so-called *extension* of a function from values to computations to a function from computations to computations, see [17] for details.

$$\begin{aligned}
\text{mfix}\ (\text{return}\cdot h) &= \text{return}\ (\text{fix}\ h) && (1)\\
\text{mfix}\ (\lambda x.a \gg\!= f\ x) &= a \gg\!= \lambda y.\text{mfix}\ (\lambda x.f\ x\ y) && (2)\\
\text{mfix}\ (\lambda\tilde{}(x,\_).\text{mfix}\ (\lambda\tilde{}(\_,y).f\ (x,y))) &= \text{mfix}\ f && (3)
\end{aligned}$$

Figure 4: Axioms for mfix. In axiom 2, x is not free in a.

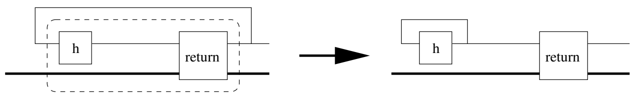


Figure 5: mfix (return $\cdot\ h$) = return (fix h)

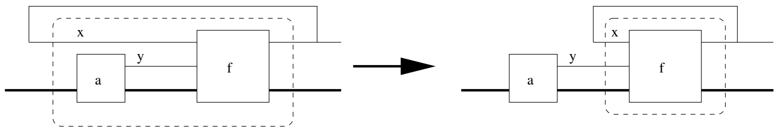


Figure 6: mfix ($\lambda x.a \gg\!= f\ x$) = $a \gg\!= \lambda y$.mfix ($\lambda x.f\ x\ y$)

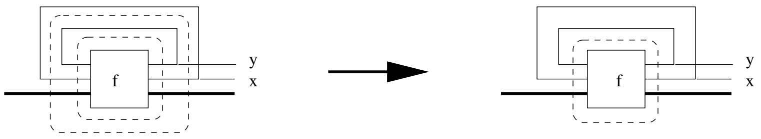


Figure 7: mfix ($\lambda\tilde{}(x,\_)$.mfix ($\lambda\tilde{}(\_,y).f\ (x,y)$)) = mfix f

If the actual computation takes place in the pure world and the result is lifted into the monad using return, the fixed-point should be the fixed-point in the pure world lifted into the monad. Figure 5 is a pictorial representation of this axiom. The dashed box represents where the mfix computation takes place. In this figure, the loop on the right hand side represents fix, while the one on the left corresponds to mfix. The thin line represents the value being processed through the computation. The thick line in the lower part of the diagram represents the computational effect (side effects, other changes in the monadic data, etc.) Notice that the fixed-point is computed only over the value part.

Axiom 2 shows how to pull a term that doesn't contribute to the fixed-point computation from the left-hand-side of a $\gg\!=$. Provided x does not appear free in a:

$$\text{mfix}\ (\lambda x.a \gg\!= f\ x) = a \gg\!= \lambda y.\text{mfix}\ (\lambda x.f\ x\ y)$$

Notice that the value of a is constant throughout the computation. Hence, we should be able to compute it only once (if need be) and put it into the fixed-point loop. Figure 6 is a pictorial representation of this axiom. Notice that both hand sides are essentially the same.

Axiom 3, depicted in Figure 7, states a useful fact about fixed-point computations involving more than one variable:

$$\text{mfix}\ (\lambda\tilde{}(x,\_).\text{mfix}\ (\lambda\tilde{}(\_,y).f\ (x,y))) = \text{mfix}\ f$$

The function f has type: $\forall a,b.\ (a,b) \rightarrow m\ (a,b)$. On the right hand side, we compute the fixed point simultaneously over both variables. On the left hand side, we perform a two step computation, where the fixed-point is computed using only one variable at a time. This axiom corresponds to Bekić's theorem for usual fixed-point computations [22]. Notice that, again, both hand sides of Figure 7 are essentially the same. It can be shown that the symmetric law:

$$\text{mfix}\ (\lambda\tilde{}(\_,y).\text{mfix}\ (\lambda\tilde{}(x,\_).f\ (x,y))) = \text{mfix}\ f$$

holds whenever axiom 3 does.

Now, we can precisely define what it means for a monad to be recursive:

Definition 1. Recursive monads. A monad m is recursive if there is a function mfix :: $\forall a.(a \rightarrow m\ a) \rightarrow m\ a$ satisfying the mfix axioms.

4.3 Derived equivalences

A direct corollary to first two mfix axioms guarantees an expected property of constant functions:

COROLLARY 1. *mfix* $(\lambda x.a) = a$, *provided x does not appear free in a.*

We also have:

COROLLARY 2. $f\ \bot \sqsubseteq$ *mfix* f

Notice that corollary 2 states more than a rudimentary fact: $f\ \bot$ yields valuable information on the structure of the fixed-point. (For instance, if $f :: a \to [a]$, and if $f\ \bot$ is a cons-cell, then so is mfix f. In particular, if $f\ \bot$ is a finite list of length k, then the length of the fixed-point is k as well.)

The polymorphic nature of mfix provides further properties. By the parametricity theorem [20], we have:

THEOREM 1. *$\forall s : A \to B,\ f : A \to m\ A,\ g : B \to m\ B$, if $g \cdot s = map\ s \cdot f$ then $map\ s\ (mfix\ f) = mfix\ g$, provided s is strict.*

As specific instances of this theorem, we obtain the following two corollaries:

COROLLARY 3. *The following equation holds for any recursive monad:*

$$\begin{aligned} &mfix\ (\lambda\tilde{\ }(x,y).f\ y \gg\!= return \cdot sp\ h\ id) \\ =\ \ &mfix\ f \gg\!= return \cdot sp\ h\ id \end{aligned} \tag{4}$$

where

$$sp\ h\ g\ z = seq\ z\ (h\ z, g\ z)$$

Notice that sp is strict in its third argument. Ignoring the strictness requirement for a moment, equation 4 becomes:

$$\begin{aligned} &\text{mfix}\ (\lambda\tilde{\ }(x,y).f\ y \gg\!= \lambda z.\text{return}\ (h\ z, z)) \\ =\ \ &\text{mfix}\ f \gg\!= \lambda z.\text{return}\ (h\ z, z) \end{aligned} \tag{5}$$

The function f only refers to y and it is fed back exactly its own result. However, the fixed-point value also gets acted upon by a *pure* function h, whose result is ignored by f. Figure 8 depicts the situation. (The symmetric case when h acts on the second component of the pair while f uses only the first component holds as well.) This equation is important because it tells us that "pure" computations that do not interfere with the fixed point computation can be performed afterwards. The strictness requirement on sp seems to be an unfortunate artifact of theorem 1; all monads that we have worked on satisfy equation 5 with no side conditions. (Unfortunately, in a framework where every type has a $\bot$ element, the parametricity theorem is weakened by the strictness requirement [12].)

Finally, we can move pure computations around:

COROLLARY 4. *Provided h is strict, the following equation holds for any recursive-monad:*

$$\begin{aligned} &mfix\ (\lambda x.f\ x \gg\!= return \cdot h) \\ =\ \ &mfix\ (\lambda x.return\ (h\ x) \gg\!= f) \gg\!= return \cdot h \end{aligned} \tag{6}$$

where $f :: a \to m\ b$ and $h : b \to a$. Equivalently:

$$mfix\ (map\ h \cdot f) = map\ h\ (mfix\ (f \cdot h))$$

Figure 9 depicts the situation. The purity requirement on h is essential: we cannot reorder any effects, as order does matter in performing them. The strictness requirement on h is quite important as well. Intuitively, the fixed-point computation on the lhs will start of by feeding $\bot$ to f, while the computation on the rhs will start of by feeding $h\ \bot$. Unless $h\ \bot = \bot$, this will provide more information to f on the rhs. Hence, we might get a $\bot$ on the lhs, while a non-$\bot$ value on the right. (We will see an example in Section 6.2.) However, there are monads for which the equality holds even when h is non-strict. The state monad is such an example (Section 6.4).

The inspiration for corollary 4 comes from a very well known law for fixed-point computations:

$$\text{fix}\ (f \cdot g) = f\ (\text{fix}\ (g \cdot f))$$

One can see the correspondence more clearly by using Kleisli composition, defined as: $f \lozenge g = \lambda x.f\ x \gg\!= g$, where x does not occur free in f or g. Now, equation 6 becomes ($\lozenge$ binds less tightly than $\cdot$):

$$\text{mfix}\ (f \lozenge \text{return} \cdot h) = \text{mfix}\ (\text{return} \cdot h \lozenge f) \gg\!= \text{return} \cdot h$$

4.4 Shrinking from right

Corollary 3 states exactly when we are allowed to pull a pure computation out from the right-hand-side of a $\gg\!=$. Can we pull out computations involving effects as well? Consider Figure 10 which depicts the case when g is allowed to make computational effects. Since the value produced by g is ignored in the fixed-point computation, one might expect pulling g out of the loop not to change the value of the computation. Indeed, our early axiomatization stipulated this property. However, it turns out that the equality is too strong for many monads. A problem arises because the monadic part of the computation in g might interfere with the fixed-point computation, possibly changing the termination behavior. Therefore, the best we could hope for is an inequality in the general case, that is:

$$\begin{aligned} &\text{mfix}\ (\lambda\tilde{\ }(x,y).f\ x \gg\!= \lambda z.g\ z \gg\!= \lambda w.\text{return}\ (z,w)) \\ \sqsubseteq\ \ &\text{mfix}\ f \gg\!= \lambda z.g\ z \gg\!= \lambda w.\text{return}\ (z,w) \end{aligned} \tag{7}$$

We will note the examples in which the equality holds or fails in Section 6.

4.5 A reflection on mfix axioms

We have tried to axiomatize how recursion over the values of monadic actions should behave. All three axioms emerge from our intuitions for recursive monadic computations. At this point, however, our approach is more definitive than explanatory in its nature, and we have felt free to work within the standard model for Haskell in which all the CPO's possess a $\bot$ element.

The extent to which our axiomatization is successful will be determined by practice. Our axioms could be deemed appropriate if they rule out useless definitions of mfix and admit only those that are meaningful in practice. Notice that we do not require a unique definition of mfix (if any) for a given monad: different applications using the same monad might conceivably benefit from different definitions of mfix. Our concern is in trying to specify the common core of monadic fixed-point computations. The major points are:

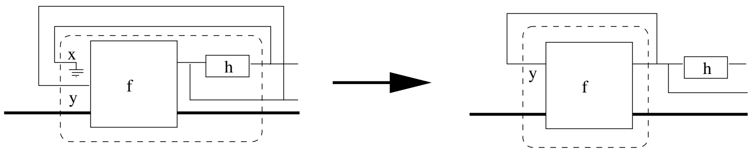


Figure 8: **mfix** $(\lambda\tilde{}(x,y).f\ y \gg\!\!= \lambda z.\mathbf{return}\ (h\ z, z)) = \mathbf{mfix}\ f \gg\!\!= \lambda z.\mathbf{return}\ (h\ z, z)$

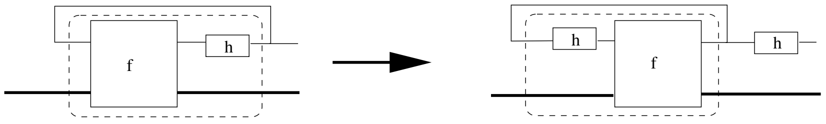


Figure 9: **mfix** $(\lambda x.f\ x \gg\!\!= \mathbf{return} \cdot h) = \mathbf{mfix}\ (\lambda x.\mathbf{return}\ (h\ x) \gg\!\!= f) \gg\!\!= \mathbf{return} \cdot h$

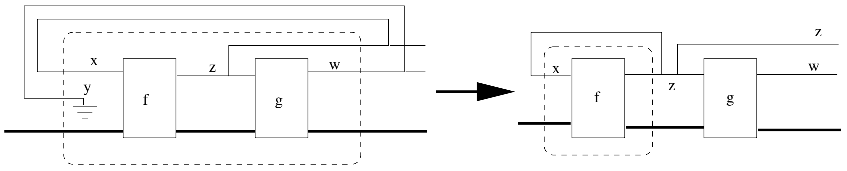


Figure 10: **mfix** $(\lambda\tilde{}(x,y).f\ x \gg\!\!= \lambda z.g\ z \gg\!\!= \lambda w.\mathbf{return}\ (z,w)) \sqsubseteq \mathbf{mfix}\ f \gg\!\!= \lambda z.g\ z \gg\!\!= \lambda w.\mathbf{return}\ (z,w)$

- The fixed-point computation should be performed only over the values.
- Effects of monadic functions should neither be duplicated nor lost in a fixed-point computation. The usual laws of "demand driven" evaluation and the structure of the underlying monad will determine when, if ever, these effects will be performed.
- In the case when there are no recursively bound variables, a μdo-expression should behave exactly like an ordinary do-expression.

Our axioms try to capture these points formally. Some monads might, of course, satisfy more laws (such as shrinking from right), and users might exploit these facts in programs. On the other hand, we believe that our axiomatization captures the minimal common core that should be satisfied by any monad in order to perform recursive computations over the results of monadic actions.

5. EMBEDDING MONADS

Whenever we want to establish that a monad is recursive, we need to prove that the axioms are satisfied by the proposed definition of mfix. In practice, we have found ourselves repeating essentially the same proof for many different monads. Recursive-monad embeddings lets us eliminate much of the duplicated work. We first recall the definition of monad homomorphisms and embeddings:

Definition 2. Monad homomorphisms/embeddings. Let m and n be two monads. A monad homomorphism, $\epsilon : m \to n$, is a family of functions (one for each type a, $\epsilon_a : m\ a \to n\ a$) such that:

$$\epsilon \cdot \mathrm{return}_m = \mathrm{return}_n \qquad (8)$$

$$\epsilon_b\ (p \gg\!\!=_m h) = \epsilon_a\ p \gg\!\!=_n \epsilon_b \cdot h \qquad (9)$$

where $p : m\ a$ and $h : a \to m\ b$. An embedding is a monic (i.e. injective) monad-homomorphism.

We extend the definition to cover the recursive case:

Definition 3. Recursive-monad homomorphisms and embeddings. Let m and n be two recursive-monads and let $\epsilon : m \to n$ be a monad homomorphism. We call ϵ a recursive-monad homomorphism if it also satisfies:

$$\epsilon\ (\mathrm{mfix}_m\ h) = \mathrm{mfix}_n\ (\epsilon \cdot h) \qquad (10)$$

Similarly, a recursive-monad embedding is a monic recursive-monad homomorphism.

We will see concrete examples of recursive-monad embeddings in the next section.

THEOREM 2. *Let $\epsilon : m \to n$ be an embedding of a monad m into a recursive-monad n. To conclude that m is recursive, it is sufficient to show that there exists a function $mfix_m$ such that ϵ is a recursive-monad embedding.*

The proof is by simple equational reasoning. We also note that equations 4, 5, 6, and 7 are preserved through monad-embeddings as well. Furthermore, composition of two embeddings is still an embedding.

This theorem not only provides a method for obtaining proofs for mfix axioms automatically for certain monads, but it also provides additional assurance that the axioms represent characteristic properties of monadic fixed-points.

6. EXAMPLES OF RECURSIVE MONADS

In this section we examine a number of monads that are frequently used in programming.

6.1 Identity

The identity monad is the monad of pure values. The Haskell declarations are:

```
newtype Id a = Id { unId :: a }

instance Monad Id where
  return x   = Id x
  Id x >>= f = f x

instance MonadRec Id where
  mfix f = fix (f . unId)
```

Notice that we use a `newtype` declaration rather than a `data`. This choice is not arbitrary. Since all Haskell data types are lifted (i.e. $\bot$ and `Id` $\bot$ are different), we would introduce an unwanted element if we had used `data`. It is a simple matter to check that mfix axioms are satisfied. One particular way of doing so is by embedding the `Id` monad into another recursive-monad, for instance the `State` monad (Section 6.4). In addition, equation 5 is satisfied, equation 6 holds even if h is non-strict, and equation 7 holds as an equality.

6.2 Maybe

The `Maybe` monad, the monad of exceptions, has the following `MonadRec` declaration:

```
instance MonadRec Maybe where
  mfix f = fix (f . unJust)
       where unJust (Just x) = x
```

The proof that the `Maybe` monad is recursive follows from the fact that it can be embedded into the `List` monad. Before studying the `List` monad, we state a lemma classifying mfix of functions for the `Maybe` monad.

Lemma 1. *The* `Maybe` *instance of mfix satisfies (*`J` *abbreviates* `Just`*,* `N` *abbreviates* `Nothing`*):*

$$\begin{array}{rcl} \mathit{mfix}\, f = \bot & \longleftrightarrow & f\ \bot = \bot \\ \mathit{mfix}\, f = \mathtt{N} & \longleftrightarrow & f\ \bot = \mathtt{N} \\ \mathit{mfix}\, f = \mathtt{J}\ \bot & \longleftrightarrow & f\ \bot = \mathtt{J}\ \bot \\ \mathit{unJust}\,(\mathit{mfix}\, f) & = & \mathit{fix}\,(\mathit{unJust} \cdot f) \end{array}$$

The first three equivalences exactly determine when the fixed-point is $\bot$, `Nothing` or `Just` $\bot$. The last equality is a consequence of the definition of mfix. An implication of these equations is that mfix of a function f of type $a \to$ Maybe a is $f\ \bot$, whenever a is a flat domain.

Equation 5 will hold for the `Maybe` monad, but a strict h is needed for equation 6. To see why, consider the following example:

```
f :: [Int] -> Maybe [Int]
f (x:_) = Just [x]

h :: [Int] -> [Int]
h xs = 1:xs
```

On the lhs of equation 6, we get $\bot$, while rhs yields `Just` $[1, 1]$. This is due to the fact that f performs a case analysis to see if its argument is a non-empty list. When the fixed-point computation starts, f first receives $\bot$ as the argument and produces $\bot$. Since $\gg\!=$ for the `Maybe` monad is strict in its first argument, the whole computation fails. On the rhs, however, f first receives $1 : \bot$, and produces `Just` $[1]$, and the computation proceeds.

We also revisit the right shrinking law (Section 4.4) within the context of the `Maybe` monad. Consider:

```
f :: [Int] -> Maybe [Int]
f xs = Just (1:xs)

g :: [Int] -> Maybe Int
g [x] = Nothing
g _   = Nothing
```

For this example, lhs of inequation 7 yields $\bot$, while the rhs yields `Nothing`. Looking closely, we see that the right hand side first produces the fixed point of f, which is of the form `Just` xs where xs is the infinite list consisting of all 1's. Then, outside the mfix loop, g ignores this value and returns `Nothing`. Within the mfix loop, the fixed-point is constructed as the limit of the chain: $\{\bot,\ 1 : \bot,\ 1 : 1 : \bot,\ \ldots\}$. When we look at the left hand side, we see a different situation. The function g acts on each value in this chain, and it yields $\bot$ for the second element. (Matching $1 : \bot$ against $[x]$ leads to nontermination.) Now, the fixed point is computed over and over starting from $\bot$, yielding $\bot$ as the result. In general, the `Maybe` monad will satisfy property 7 as an inequality. If we look more closely, we see that the problem lies within the fact that $\gg\!=$ for the `Maybe` monad is strict in its first argument. Unfortunately, there is no way to alleviate this problem. We conclude that this property can not be satisfied as long as the $\gg\!=$ of the monad is strict in its first argument. This requirement practically rules out any data type that has more than one constructor from satisfying property 7 as an equality.

6.3 List

The `MonadRec` declaration for the `List` monad is:

```
instance MonadRec [] where
  mfix f = case fix (f . head) of
             []    -> []
             (x:_) -> x : mfix (tail . f)
```

The intuition behind this definition of mfix is the following: For a function of type $a \to [a]$, the fixed point is of type $[a]$, i.e. it's a list. Each element of this fixed-point should be the fixed point of the function restricted to that particular position. That is, the ith entry of the fixed point of a function with type $a \to [a]$, say f, should be the fixed point of the function: $\mathrm{head} \cdot \mathrm{tail}^i \cdot f$. In other words,

$$\mathrm{mfix}\ (\lambda x.[h_1\ x, \ldots, h_n\ x]) = [\mathrm{fix}\ h_1, \ldots, \mathrm{fix}\ h_n]$$

or, more generally:

$$\mathrm{mfix}\ f = \mathrm{fix}\ (\mathrm{head} \cdot f) : \mathrm{mfix}\ (\mathrm{tail} \cdot f)$$

This definition will work well as long as the fixed-point is an infinite list. However, it fails to capture the finite case. Notice that we are computing the fixed points of the functions of the form $\mathrm{head} \cdot f$. If f ever returns `[]`, we want to stop the computation, rather than taking the head (which will yield $\perp$). Hence, recalling that

$$\mathrm{fix}\ (\mathrm{head} \cdot f) = \mathrm{head}\ (\mathrm{fix}\ (f \cdot \mathrm{head}))$$

we can compute the fixed points of the functions of the form $f \cdot \mathrm{head}$ (whose results will be lists), and stop when we get an empty list. Putting these ideas together, we arrive at the definition we have given above.

Analogous to Lemma 1, we have:

LEMMA 2. *The* `List` *instance of mfix satisfies:*

$$\begin{array}{rcl}
\mathit{mfix}\ f = \perp & \longleftrightarrow & f\ \perp = \perp \\
\mathit{mfix}\ f = [\,] & \longleftrightarrow & f\ \perp = [\,] \\
\mathit{mfix}\ f = [\perp] & \longleftrightarrow & f\ \perp = [\perp] \\
\mathit{head}\ (\mathit{mfix}\ f) & = & \mathit{fix}\ (\mathit{head} \cdot f) \\
\mathit{tail}\ (\mathit{mfix}\ f) & = & \mathit{mfix}\ (\mathit{tail} \cdot f) \\
\mathit{mfix}\ (\lambda x.f\ x : g\ x) & = & \mathit{fix}\ f : \mathit{mfix}\ g \\
\mathit{mfix}\ (\lambda x.f\ x +\!\!+\ g\ x) & = & \mathit{mfix}\ f +\!\!+\ \mathit{mfix}\ g
\end{array}$$

The first two equivalences imply that when $f\ \perp$ is a cons-cell (i.e. of the form $(x : xs)$), then mfix f is also a cons-cell. Using this lemma, proving that mfix axioms hold is a tedious but straightforward exercise.

The embedding of the `Maybe` monad into the `List` monad simply takes `Nothing` to `[]` and `Just x` to [x]. By the discussion on embeddings, we do not expect the `List` monad to satisfy equation 6 when h is non-strict and equation 7 as an equality since we know that the `Maybe` monad does not have these properties. In deed, it is possible to construct counterexamples for the `List` monad as well. Equation 5, as with all other cases, holds for the `List` monad.

We can finally solve the puzzle posed in the introduction. First, using our intuition for the `List` monad and recursive bindings, we try to derive the solution. It is well known that the do-notation and the usual list comprehensions of Haskell coincide for the `List` monad. Hence, we can think of the puzzle as the following list comprehension:

```
[x+y | (x, z) <- [(y, 1), (y^2, 2), (y^3, 3)],
       Just y <- map isEven [z+1 .. 2*z]]
```

Notice that this list comprehension is still not valid Haskell: The variable y is used before its value is generated. Nevertheless, we apply the usual rules for decomposing list comprehensions [21]. We obtain:

```
concat [   [x+y | Just y <- map isEven [z+1 .. 2*z]]
         | (x, z) <- [(y, 1), (y^2, 2), (y^3, 3)] ]
```

Notice that we have a nested comprehension now. At this point, we can expand the outer comprehension for each assignment to (x, z). This step is where we use our intuition for the recursive bindings for interpreting the free variable y: We substitute it for x symbolically. This yields:

```
concat [ [y  +y | Just y <- map isEven [2 .. 2]],
         [y^2+y | Just y <- map isEven [3 .. 4]],
         [y^3+y | Just y <- map isEven [4 .. 6]] ]
```

Now, routine calculations yield: `[4, 20, 68, 222]`.

When we run the puzzle using the μdo modifed version of Hugs (after replacing the keyword `do` with `mdo`), we get exactly the same answer.

6.4 State

The `State` monad is used to capture computations that involve mutable variables [13, 14]. Here are the definitions:

```
newtype State s a = ST { unST :: (s -> (a, s)) }

instance Monad (State s) where
  return x   = ST (\s -> (x, s))
  ST f >>= g = ST (\s -> let (a, s') = f s
                         in unST (g a) s')

instance MonadRec (State s) where
  mfix f = ST (\s -> let (a, s') = unST (f a) s
                     in (a, s'))
```

Without tags, the definition of mfix is simply:

$$\mathrm{mfix}\ f = \lambda s.\mathrm{let}\ (a, s') = f\ a\ s\ \mathrm{in}\ (a, s')$$

The `State` monad satisfies all mfix axioms, hence it is recursive. The definition of mfix clearly shows that the fixed-point computation is performed only on values, the state component is left untouched. Furthermore, equation 5 holds, equation 6 does not require a strict h and property 7 is satisfied as an equality.

6.5 State with exceptions

Often the computations that have side effects fail to yield a value. This concept is generally modeled with a combination of the state and exception monads. In this section we look at two examples.

The first version considers the case when neither a value nor an updated state might be available after a computation. The declarations are (again, we drop explicit tags):

```
newtype STE s a = s -> Maybe (a, s)

instance Monad (STE s) where
  return x = \s -> Just (x, s)
  f >>= g  = \s -> case f s of
                     Nothing      -> Nothing
                     Just (a, s') -> g a s'
```

```
instance MonadRec (STE s) where
  mfix f = \s -> let a = f b s
                     b = fst (unJust a)
                 in a
```

Now we consider when the computation might fail but an updated state is still available. The declarations are:

```
newtype STE2 s a = s -> (Maybe a, s)

instance Monad (STE2 s) where
  return x = \s -> (Just x, s)
  f >>= g  = \s -> case f s of
                   (Nothing, s') -> (Nothing, s')
                   (Just a, s')  -> g a s'

instance MonadRec (STE2 s) where
  mfix f = \s -> let a = f b s
                     b = unJust (fst a)
                 in a
```

In both cases, the computation of the fixed-point is similar to those of `State` and `Maybe` monads. We equate the value part of the result with the input to the function. Notice the symmetry between the definitions and the `newtype` declarations.

It turns out both of these monads are recursive. However, they require strict h for satisfying equation 6 and they don't satisfy equation 7 as an equality. This is hardly surprising since the `Maybe` monad behaves like this as well. As with all other cases, both monads satisfy equation 5.

6.6 Other monads

We take a brief look at a couple of other monads without going into much detail. The `Reader` (or *environment*) monad is a version of the `State` monad where we only read the state without ever changing it [9]. The obvious embedding into the `State` monad suffices to prove that the `Reader` monad is recursive. In fact, the work on implicit parameters [15] provides an *implicit* recursive `Reader` monad in Haskell where the usual `let` construct expresses recursive computations, (implicit parameters provide the mechanism for accessing values within the monad).

The output monad, as described in Section 2.1, is recursive also. It also embeds into the `State` monad. The definition of mfix is:

```
instance MonadRec Out where
  mfix f = fix (f . unOut)
        where unOut (Out (a, _)) = a
```

The `Tree` monad [9] is recursive. The definition of mfix closely mimics that of the `List` monad. Unlike lists, however, these trees are never empty and hence the `List` monad cannot be embedded into them. It is, however, not clear what sort of applications can benefit from recursive bindings for the `Tree` monad.

Two other recursive-monads that are very well known in the Haskell community are the internal input/output (`IO`) and state (`ST`) monads. The library functions `fixIO` and `fixST` correspond to our mfix for the `IO` and `ST` monads, respectively. These monads are "internal" in the sense that, unlike others, their implementations use destructive updates and hence need to be defined as primitives. This prevents us from constructing explicit proofs, but the Haskell folklore suggests that they indeed satisfy our axioms.

The details on these monads (along with other technical development) can be found in [5].

The continuation monad continues to cause us grief. We have been unable to produce a viable definition for mfix in this case. Furthermore, in the Scheme case—when state and continuations coexist—the semantics implied by the definition of `letrec` (Scheme's equivalent of mfix) seems to lack the appropriate uniformity properties implied by axiom 2 and property 7.

7. AN EXAMPLE: DOUBLY LINKED CIRCULAR LISTS

In this section, we give a hands-on example of using monadic recursive bindings. We will create a doubly linked circular list which can be traversed forwards or backwards. At each node, we will store a flag indicating whether the node has been visited before. Since this flag needs to be mutable, we will use the internal `IO` monad to get access to the required reference cells. Obviously, this example can be generalized to any problem where we have interlinked stateful objects with possibly cyclic dependencies.

We first import the `MonadRec` and `IOExts` libraries. The first one declares the `MonadRec` class. The `IOExts` library provides `fixIO`, as discussed above, along with functions to create and manipulate mutable variables:

```
> import MonadRec
> import IOExts
```

The `MonadRec` declaration for IO is trivial:

```
> instance MonadRec IO where
>   mfix = fixIO
```

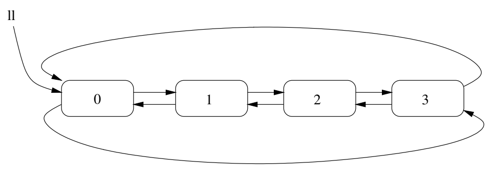
By this declaration, the μdo-notation becomes available for the `IO` monad. Each node in our list will have a mutable boolean value indicating whether it has been visited, left and right nodes and a single integer value for the data:

```
> data N = N (IORef Bool, N, Int, N)
```

To create a new node with value `i` in between the nodes `b` and `f`, we use the function `newNode`:

```
> newNode :: N -> Int -> N -> IO N
> newNode b i f = do v  <- newIORef False
>                    return (N (v, b, i, f))
```

Notice that the visited flag is set to `False`. We will use this function to create the following structure:



Here's the code for it:

```
> ll = mdo n0 <- newNode n3 0 n1
>          n1 <- newNode n0 1 n2
>          n2 <- newNode n1 2 n3
>          n3 <- newNode n2 3 n0
>          return n0
```

The use of μdo is essential: the cyclic nature of the construction is not expressible using an ordinary do-expression. We can test our implementation with a traversal function:

```
> data Dir = F | B deriving Eq
>
> traverse :: Dir -> N -> IO [Int]
> traverse d (N (v, b, i, f)) =
>     do visited <- readIORef v
>        if visited
>           then return []
>           else do writeIORef v True
>                   let next = if d == F then f else b
>                   is <- traverse d next
>                   return (i:is)
```

Here's a sample run:

```
Main> ll >>= traverse F >>= print
[0,1,2,3]
Main> ll >>= traverse B >>= print
[0,3,2,1]
```

The inverse function that takes a non-empty list and constructs a doubly linked circular list out of its elements illustrates the use of μdo further:

```
> l2dll :: [Int] -> IO N
> l2dll (x:xs) = mdo c      <- newNode l x f
>                    (f, l) <- l2dll' c xs
>                    return c
>
> l2dll' :: N -> [Int] -> IO (N, N)
> l2dll' p []     = return (p, p)
> l2dll' p (x:xs) = mdo c      <- newNode p x f
>                       (f, l) <- l2dll' c xs
>                       return (c, l)
```

Note in particular the essential use of μdo in the construction of the linked list.

8. RELATED WORK

There is a major line of research that attempts to characterize fixed points in general [3, 7, 19]. This work has only a passing relevance to the work here. Haskell already has one brand of monadic fixed points—those obtained when writing recursive monadic functions by using Haskell's built in recursion—and these are the fixed points picked out by this general work. As we indicated in Section 4.1, this generic fixed point is not able to achieve recursive bindings in the way we want. Instead, we have had to describe a value-recursion that does not repeat the monadic effect.

Much greater similarity to our work is found in O'Haskell, which is is a concurrent, object oriented extension of Haskell designed for addressing issues in reactive functional programming [18]. One application of O'Haskell is in programming layered network protocols. Each layer interacts with its predecessor and successor by receiving and passing information in both directions. In order to connect two protocols that have mutual dependencies, one needs a recursive knot-tying operation. Since O'Haskell objects are monadic, recursive monads are employed in establishing connections between objects. O'Haskell adds a keyword `fix` to the do-notation whose translation is a simplified version of ours. The O'Haskell work, however, does not try to axiomatize or generalize the idea any further.

Although we limited our attention to monadic computations, recursion makes sense in the more general setting of *arrows* as well [8]. Recently, Ross Paterson axiomatized `arrowFix`, the arrow version of mfix, which turns out to be quite similar to our formulation. He echoes aspects of Hasegawa's work in traced monoidal categories that provides a general framework for recursion and cyclic sharing [7]. Again, because a different notion of fixed point is required, Paterson relaxed some of Hasegawa's axioms, and replaced others completely. (Unfortunately, Paterson's work is not published yet.)

Recent work by Friedman and Sabry tries to address the problem from a different angle [6]. Rather than an axiomatization, their work suggests combining monads with a state monad and performing a generic recursion computation in this combined world. The semantics of recursion is then defined by this implementation. Since the recursion is performed in the combined monad, it is the users responsibility to translate original problems and values to and from this combined world.

Proofs of the claims made in this paper and other technical details are reported in a technical report [5]. A more detailed treatment of the translation of μdo-expressions appear in a companion implementation paper [4]. The web page `http://www.cse.ogi.edu/PacSoft/projects/muHugs` contains the software, papers and other research material related to this work.

9. CONCLUSIONS

Monads play an important role in functional programming by providing a clean methodology for expressing computational effects. Monadic computations use a certain sublanguage shaped by the functions that act on monadic objects. Haskell makes this approach quite convenient by providing the do-notation. A shortcoming, however, is that recursion over the results of monadic actions can not be conveniently expressed. Furthermore, it is not clear how to perform recursion on values in the presence of effects. In order to alleviate this problem, we have axiomatized monadic fix and implemented an extension to the do-notation, which can be used in expressing such recursive computations in a natural way. We expect that many applications can benefit from this work, as monads become more pervasive in functional programming.

Even though we have proposed a separate μdo construct, we believe that the usual do-expression of Haskell should be extended to capture this new style of programming. That is, there should not be a separate μdo keyword, but rather the compiler should analyze do-expressions to see if recursive bindings are employed, performing the translations as appropriate. An ambitious compiler may also perform simplifications based on the mfix axioms.

10. ACKNOWLEDGEMENTS

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